Reinforcement Learning

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Lecture 6: Deep and Robust Reinforcement Learning

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License Information for Reinforcement Learning (EE-568)

Recap: Overview of reinforcement learning approaches

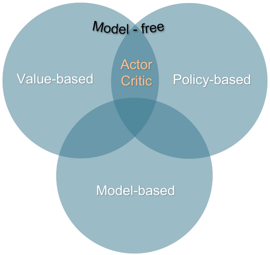
Value-based RL (Critic-only)

◦ Learn the optimal value functions $V^\star, Q^\star$

◦ Algorithms: Monte Carlo, SARSA, Q-learning, etc.

◦ Use temporal difference (low variance)

◦ Does not scale to large action spaces



Policy-based RL (Actor-only)

◦ Learn the optimal policy via gradient methods

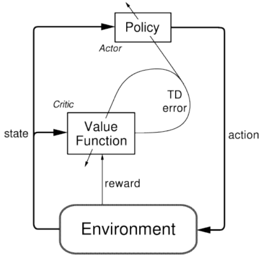
◦ Algorithms: PG, NPG, TRPO, PPO, etc.

◦ Scales to large or continuous action spaces

◦ High variance, sample inefficiency

Actor-critic (AC) methods

◦ AC methods aim at combining the advantages of actor-only methods and critic-only methods.



Interaction of Actor-critic [29].

◦ The actor uses the policy gradient to update the learning policy.

◦ The critic uses TD learning to estimate the value function.

Actor-critic methods

◦ Actor-critic algorithms follow an approximate policy gradient:

$$\nabla_\theta J(\pi_\theta) \approx \frac{1}{1-\gamma}\mathbb{E}_{s\sim\lambda_\mu^{\pi_\theta}}\left[\mathbb{E}_{a\sim\pi_\theta(\cdot|s)}\left[Q_w(s,a)\nabla_\theta \log \pi_\theta(a \mid s)\right]\right].$$

$$\nabla_\theta J(\pi_\theta) \approx \frac{1}{1-\gamma}\mathbb{E}_{s\sim\lambda_\mu^{\pi_\theta}}\left[\mathbb{E}_{a\sim\pi_\theta(\cdot|s)}\left[A_w(s,a)\nabla_\theta \log \pi_\theta(a \mid s)\right]\right].$$

◦ Actor: adjust the policy parameter θ using policy gradient using the value function estimated by the critic.

◦ Critic: update the parameter w to estimate action-value or advantage function.

$$Q_w(s,a) \approx Q^{\pi_\theta}(s,a)$$
$$A_w(s,a) \approx Q^{\pi_\theta}(s,a) - V^{\pi_\theta}(s)$$

Bias in actor-critic methods

- Recall action value expression of policy gradient

$$\nabla_\theta J(\pi_\theta) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim\lambda_\mu^{\pi_\theta}}\left[\mathbb{E}_{a\sim\pi_\theta(\cdot|s)}\left[Q^{\pi_\theta}(s,a)\nabla_\theta \log \pi_\theta(a \mid s)\right]\right].$$

- Policy gradient estimators used by actor-critic algorithms:

$$\hat{\nabla}_\theta J(\pi_\theta) = \frac{1}{1-\gamma}\mathbb{E}_{s\sim\lambda_\mu^{\pi_\theta}}\left[\mathbb{E}_{a\sim\pi_\theta(\cdot|s)}\left[Q_w(s,a)\nabla_\theta \log \pi_\theta(a \mid s)\right]\right].$$

- Approximating the policy gradient using value function approximation Q_w could introduce bias.

- Luckily, if the value function approximation Q_w is chosen carefully, one may avoid such bias.

Compatible function approximation theorem

Compatible function approximation theorem [30]

Suppose the following two conditions are satisfied:

- Value function approximation at $w^\star$ is compatible to the policy, i.e.,

$$\nabla_w Q_{w^\star}(s,a) \;=\; \nabla_\theta \log \pi_\theta(a \mid s).$$

- Value function parameter $w^\star$ minimizes the mean-squared error, i.e.,

$$\min_w \; \mathbb{E}_{s\sim\lambda_\mu^{\pi_\theta}, a\sim\pi_\theta(\cdot|s)}[(Q_w(s,a) - Q^{\pi_\theta}(s,a))^2].$$

Then the policy gradient using critic $Q_{w^\star}(s,a)$ is exact:

$$\nabla_\theta J(\theta) \;=\; \frac{1}{1-\gamma}\mathbb{E}_{s\sim\lambda_\mu^{\pi_\theta}, a\sim\pi_\theta(\cdot|s)}[\nabla_\theta \log \pi_\theta(a \mid s) Q_{w^\star}(s,a)].$$

Remarks:

- Proof follows immediately from first-order optimality condition.
- Example: $Q_w(s,a) = \nabla_\theta \log \pi_\theta(a \mid s)^\top w$.

Variant I: Online action-value actor-critic

Online Action-Value Actor-critic Algorithm

Initialize θ_0, w_0, state $s_0 \sim \mu$, $a_0 \sim \pi_{\theta_0}(\cdot \mid s_0)$.

for each step of the episode $t = 0, ..., T$ **do**

Obtain (r_t, s_{t+1}, a_{t+1}) from π_{θ_t}.

Compute policy gradient estimator: $\hat{\nabla}_\theta J(\pi_{\theta_t}) = Q_{w_t}(s_t, a_t) \nabla_\theta \log \pi_{\theta_t}(a_t \mid s_t)$.

Actor update θ: $\theta_{t+1} = \theta_t + \alpha_t \hat{\nabla}_\theta J(\pi_{\theta_t})$.

Compute temporal difference: $\delta_t = r_t + \gamma Q_{w_t}(s_{t+1}, a_{t+1}) - Q_{w_t}(s_t, a_t)$.

Critic update: $w_{t+1} = w_t - \beta_t \delta_t \nabla_w Q_{w_t}(s_t, a_t)$.

end for

Remarks:

- Uses temporal difference to estimate the value function Q^{π_θ}.
- Examples for Q_w: linear value function approximation $Q_w(s, a) = \phi(s, a)^\top w$.

Variant II: Advantage actor-critic

Advantage Actor-critic (A2C)

Initialize θ_0, w_0, state $s_0 \sim \mu$.

for each step of the episode $t = 0, ..., T$ **do**

Take action $a_t \sim \pi_{\theta_t}(\cdot \mid s_t)$, obtain (r_t, s_{t+1}).

Estimate advantage function: $\delta_t = r_t + \gamma V_{w_t}(s_{t+1}) - V_{w_t}(s_t)$.

Compute policy gradient estimator: $\hat{\nabla}_\theta J(\pi_{\theta_t}) = \delta_t \nabla_\theta \log \pi_{\theta_t}(a_t \mid s_t)$.

Actor update: $\theta_{t+1} = \theta_t + \alpha_t \hat{\nabla}_\theta J(\pi_{\theta_t})$.

Critic update: $w_{t+1} = w_t - \beta_t \delta_t \nabla_w V_{w_t}(s_t)$.

end for

Remarks:

- Use $V_w(s)$ to approximate $V^{\pi_\theta}(s)$, for instance $V^w(s) \approx \phi(s)^\top w$.
- Use one step lookahead to estimate $Q^{\pi_\theta}(s_t, a_t) \approx r(s_t, a_t) + \gamma V^{\pi_\theta}(s_{t+1})$.
- Use advantage function to approximate the policy gradient.

Various actor-critic extensions

- Natural actor-critic [21]: use TRPO[26] or NPG[12] to update the actor
- Actor-critic with generalized advantage estimator [27]: generalize advantage function with TD(λ)

$$\hat{A}_t^k(s_t, a_t) = r(s_t, a_t) + \gamma r(s_{t+1}, a_{t+1}) + \cdots + \gamma^k V_w(s_{t+k}) - V_w(s_t)$$

$$\hat{A}_t^{\mathrm{GAE}}(s_t, a_t) = (1-\lambda)\sum_{k=1}^{\infty} \lambda^{k-1} \hat{A}_t^k(s_t, a_t)$$

- Soft actor-critic [9]: use entropy regularization in the objective to improve exploration

$$\max_\pi \mathbb{E}\left[\sum_{t=0}^{\infty} \gamma^t r(s_t, a_t) + \lambda \cdot \mathcal{H}(\pi(\cdot|s_t))\right], \text{ where } \mathcal{H}(\pi(\cdot|s)) = \mathbb{E}_{a\sim\pi(\cdot|s)}[-\log \pi(a|s)]$$

Convergence of actor-critic methods

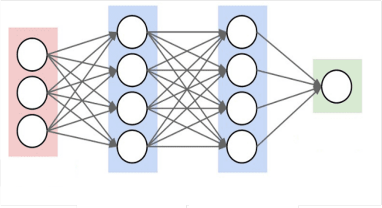
Remarks:

- ∘ There is an asymptotic analysis of two time-scale actor-critic methods (i.e., $\lim_{t\to\infty} \frac{\alpha_t}{\beta_t} = 0$) [3, 13].
- ∘ The proof is based on two-time-scale stochastic approximation and ODE analysis.
- ∘ Finite-sample analyses of actor-critic methods (tabular or LFA) have been studied very recently.
- ∘ This work is based on the bilevel optimization perspective; see e.g., [38].
- ∘ Indeed, actor-critic algorithms can be formulated as bilevel optimization:

$$\min_{\theta} \; F(\theta) = f(\theta, w^\star(\theta)), \quad \text{(Upper level)}$$
$$\text{s.t. } w^\star(\theta) \in \arg\min_{w} \ell(\theta, w). \quad \text{(Lower level)}$$

Deep reinforcement learning = DL + RL

- Tabular methods and linear function approximation are insufficient for large-scale RL applications.
- Using neural networks seems to be a must.



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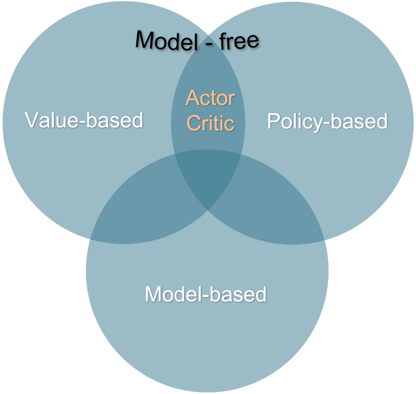


Neural networks

- Nested composition of (learnable) linear transformation with (fixed) nonlinear activation functions
- Example: a single-layer neural network (shallow neural network)

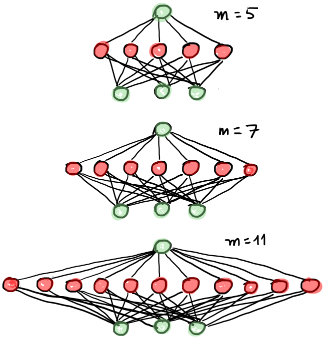


Figure: Networks of increasing width

$$f(\mathbf{x}; W, \boldsymbol{\alpha}) = \sum_{i=1}^{m} \alpha_i \cdot \sigma(w_i^\top \mathbf{x})$$

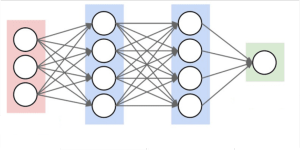
Activation function $\sigma(\cdot)$

- Identity: $\sigma(u) = u$
- Sigmoid: $\sigma(u) = \frac{1}{1+\exp(-u)}$
- Tanh: $\sigma(u) = \tanh(u)$
- Rectified linear unit (ReLU): $\sigma(u) = \max(0, u)$
- ...

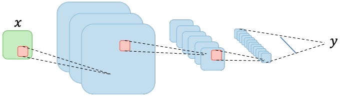
Deep neural networks

◦ More hidden layers, different activation functions, more general graph structure ....

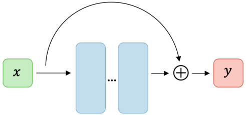
Feed forward network



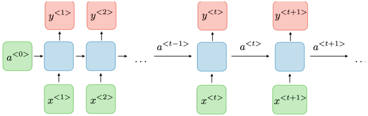
Convolutional network



Residual network



Recurrent network



Why neural networks?

- Universal approximation:
  - Any continuous function on a compact domain can be (uniformly) approximated to arbitrary accuracy
  - A single-hidden layer neural network suffices with a non-polynomial activation function.
  - Classical references include [4, 11, 1].
  - However, the number of neurons can be extremely large to approximate *any* continuous function.

- Benefits of depth:
  - A deep network cannot be approximated by a reasonably-sized shallow network [39]
  - A function with $O(L^2)$ layers and width 2 requires width $O(2^L)$ to approximate with $O(L)$ layers [31].
  - For more refined depth separation results see [24].

Example: ATARI network architecture

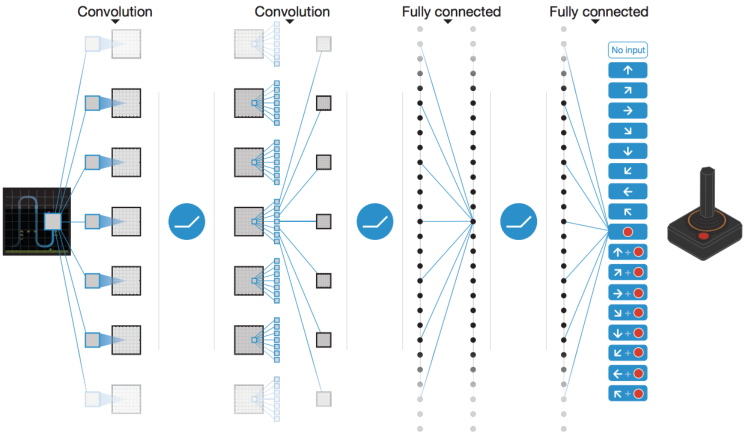


Figure: ATARI Network Architecture for $Q(s, a)$: History of frames as input. One output per action. [18]

Challenges with training neural networks in RL

- Deadly triad (i.e., divergence when combining function approximation, bootstrapping, and off-policy learning)
- Non iid data
- Sample inefficiency
- High variance
- Overfitting
- Saddle points
- ...

Common fixes or RL tricks

- Better data: e.g., experience replay (mix online data and a buffer from past experience)
  - Reduce correlation, allow mini-batch update
- Better objective: e.g., use entropy regularization
  - Improve optimization landscape, encourage exploration
- Better optimizers: e.g., adaptive SGD such as Adam and RMSProp
  - Adaptive learning rates
- Better estimation: e.g., Use eligibility traces, target works
  - Reduce overestimation bias, balance bias-variance tradeoff
- Better sampling: e.g., use prioritized replay (sample based on priority)
  - Prioritize transitions on which we can learn much
- Better implementation: e.g., parallel implementation (multithreading of CPU)
  - Speed up training, reduce correlation, allow better exploration
- Better architectures: e.g. dueling networks
  - Encode inductive biases that are good for RL

Value-based DRL

- Idea: Use neural networks for value function approximation
- Recall Q-learning:

Q Learning

$$Q(s_t, a_t) \leftarrow Q(s_t, a_t) + \alpha_t [r_t + \gamma \max_a Q(s_{t+1}, a) - Q(s_t, a_t)]$$

Q-learning with function approximation

$$w_{t+1} \leftarrow w_t + \alpha_t [r_t + \gamma \max_a Q_{w_t}(s_{t+1}, a) - Q_{w_t}(s_t, a_t)] \nabla Q_{w_t}(s_t, a_t)$$

Remarks:

- Note that Q-learning is not an unbiased stochastic gradient descent method.
- Naive deep Q-learning could diverge due to sample correlation and moving targets.
- Deep Q-networks [18]: combine several techniques for stabilizing Q-learning.
  - Experience replay (better data efficiency and make data more stationary).
  - Target networks (prevent target objective from changing too fast).
- Experience replay (better data efficiency and make data more stationary).
- Target networks (prevent target objective from changing too fast).

Deep Q-Networks (DQN)

- Main idea: minimize the following mean-square error by SGD (or adaptive SGD)

$$\min_{w} L(w) = \mathbb{E}_{s,a,r,s' \sim \mathcal{D}} \left[\left(r + \gamma \max_{a'} Q(s', a'; w^-) - Q(s, a; w) \right)^2 \right]$$

- The target parameter w^- is held fixed and updated periodically

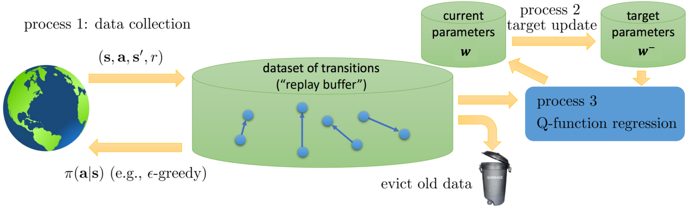


Figure: A more general view of DQN. Source: `https://zhuanlan.zhihu.com/p/468385820`

DQN in playing Atari games [18]

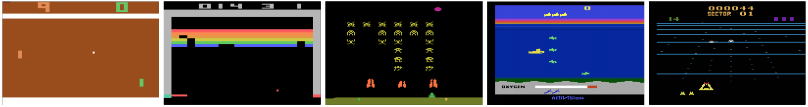


Figure: Five Atari 2600 Games: Pong, Breakout, Space Invaders, Seaquest, Beam Rider

	B. Rider	Breakout	Enduro	Pong	Q*bert	Seaquest	S. Invaders
Random	354	1.2	0	−20.4	157	110	179
Sarsa [3]	996	5.2	129	−19	614	665	271
Contingency [4]	1743	6	159	−17	960	723	268
DQN	**4092**	**168**	**470**	**20**	**1952**	**1705**	**581**
Human	7456	31	368	−3	18900	28010	3690

Figure: Average total reward for a fixed number of steps.

∘ DQN source code: https://github.com/deepmind/dqn

DQN extensions I

- Double DQN [33] uses separate networks to select best action
- It then evaluates best action to reduce overestimation bias

$$\min_{w} L(w) = \mathbb{E}_{s,a,r,s' \sim \mathcal{D}} \left[\left(r + \gamma Q(s', \arg\max_{a'} Q(s', a'; w); w^{-}) - Q(s, a; w) \right)^2 \right]$$

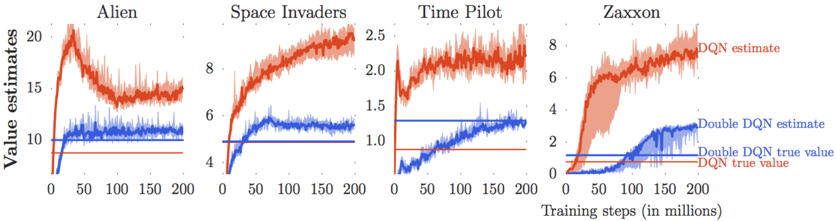


Figure: Value estimates by DQN (orange) and Double DQN (blue) on Atari games. The straight horizontal lines are computed by running the corresponding agents after learning concluded, and averaging the actual discounted return obtained from each visited state.

DQN extensions II

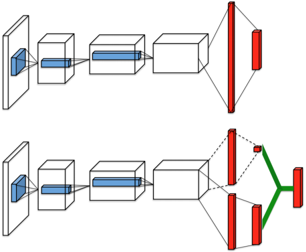
∘ DQN with prioritized experience replay [25]: Prioritize transitions in proportion to the absolute Bellman error

$$p \propto \left| r + \gamma \max_{a'} Q(s', a'; w) - Q(s, a; w) \right|$$

$\langle S_t, A_t, R_{t+1}, S_{t+1}, p_t \rangle$
$\langle S_{t+1}, A_{t+1}, R_{t+2}, S_{t+2}, p_{t+1} \rangle$
$\langle S_{t+2}, A_{t+2}, R_{t+3}, S_{t+3}, p_{t+2} \rangle$
$\langle S_{t+3}, A_{t+3}, R_{t+4}, S_{t+4}, p_{t+3} \rangle$
$\ldots$

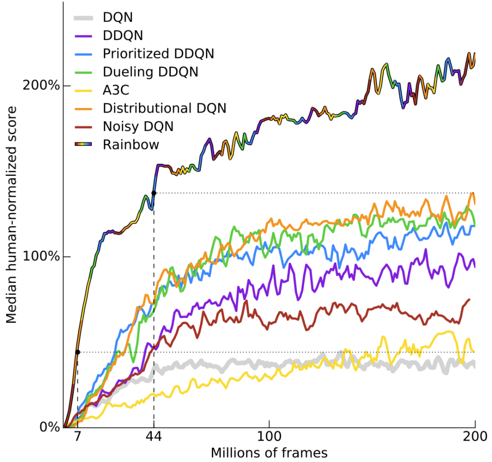
∘ Dueling DQN [35]: Split Q-networks into two streams to estimate value function and advantage function

$$Q(s, a; w, \alpha, \beta) \;=\; V(s; w, \beta) + \bar{A}(s, a; w, \alpha)$$



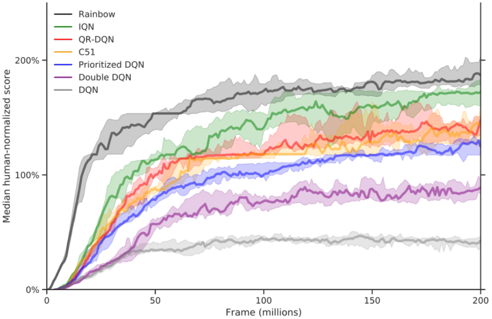
DQN mega extension

- Can these extensions be combined? Yes, Rainbow [10]!



The big zoo of DQN

Directory	Paper
dqn	Human Level Control Through Deep Reinforcement Learning
double_q	Deep Reinforcement Learning with Double Q-learning
prioritized	Prioritized Experience Replay
c51	A Distributional Perspective on Reinforcement Learning
qrdqn	Distributional Reinforcement Learning with Quantile Regression
rainbow	Rainbow: Combining Improvements in Deep Reinforcement Learning
iqn	Implicit Quantile Networks for Distributional Reinforcement Learning



Plot of median human-normalized score over all 57 Atari games for each agent

◦ Source code: https://github.com/deepmind/dqn_zoo

Policy-based/Actor-critic DRL

- Combine the actor-critic approach with Deep Q Network
  - Asynchronous advantage actor-critic (A3C)) [17]
  - Soft actor critic (SAC) [9]
  - Deep deterministic policy gradient (DDPG) [16]: continuous control
  - Twin delayed DDPG (TD3) [7]: continuous control
  - ....

A3C [17]

◦ Idea: advantage actor-critic + deep Q-network + asynchronous implementation

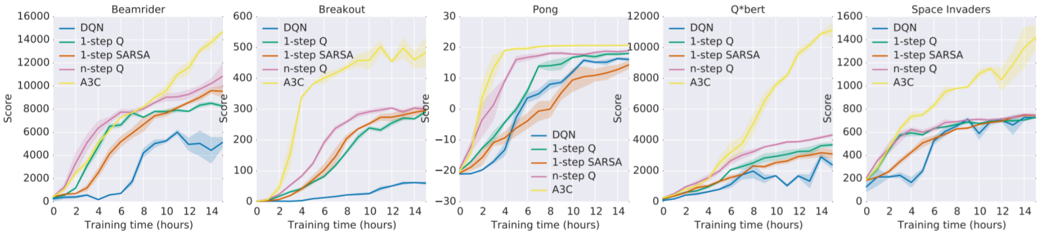


Figure: Comparison for DQN and A3C on five Atari 2600 games. 1-step Q means asynchronous one-step Q-learning.

DDPG [16] and TD3 [7]

- DDPG: deterministic policy gradient + deep Q-network
- Select action $a \sim \mu(s;\theta) + \mathcal{N}(0, \sigma^2)$ (add noise to enhance exploration)
- Policy update: $\nabla_\theta J(\theta) \approx \frac{1}{N} \sum_i \nabla_a Q_w(s_i, \mu(s_i;\theta)) \nabla_\theta \mu(s_i;\theta)$
- TD3: DDPG + clipped action exploration + delayed policy update + pessimistic double Q-learning
  - Select action $a \sim \mu(s;\theta) + \epsilon$, $\epsilon \sim \text{clip}(N(0, \sigma^2), -c, c)$
  - Delayed policy update: update critic more frequent than policy

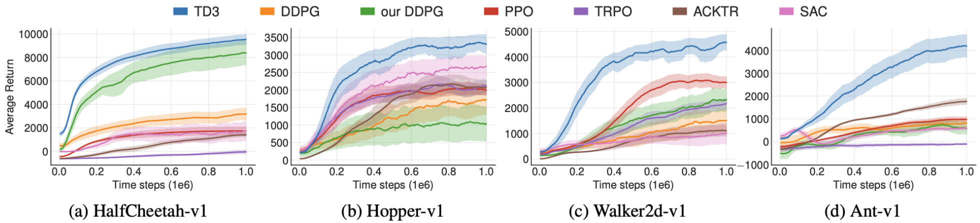


Figure: Learning curves for the OpenAI gym continuous control tasks.

Summary

- **Deep Value-based Methods**
  - DQN
  - Double DQN
  - Dueling DQN
  - DQN with prioritized experience replay
  - Rainbow
  - ....

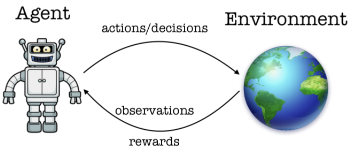
- **Deep Policy-based/Actor-critic Methods**
  - TRPO
  - PPO
  - A3C
  - SAC
  - DDPG/TD3
  - ....

Question: So, which one should we choose in practice? when do they work well?

Deep RL resources

- OpenAI Spinning up: https://spinningup.openai.com/
- The awesome list of deep RL (libraries and tutorials): https://github.com/kengz/awesome-deep-rl

Reinforcement learning



- Environment: Markov Decision Process (MDP) $\mathcal{M} = (\mathcal{S}, \mathcal{A}, T, \gamma, \mu, r)$
- Agent: Parameterized deterministic policy $\pi_\theta : \mathcal{S} \to \mathcal{A}$, where $\theta \in \Theta$

Reinforcement learning (RL) game

At time step $t = 0$: $S_0 \sim \mu(\cdot)$

for $t = 1, 2, \ldots$ **do**:

  agent observes the environment's state $S_t \in \mathcal{S}$

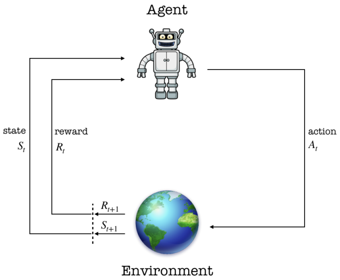
  agent chooses an action $A_t = \pi_\theta(S_t) \in \mathcal{A}$

  agent receives a reward $R_{t+1} = r(S_t, A_t)$

  agent finds itself in a new state $S_{t+1} \sim T(\cdot \mid S_t, A_t)$

Exploration vs. exploitation in RL

- Challenge: Exploration vs. exploitation!



- Objective (non-concave): $$\max_{\theta \in \Theta} \; J(\theta) \; := \; \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} R_t \;\middle|\; \pi_\theta, \mathcal{M}\right]$$
  - The environment only reveals the rewards after actions
  - Exploitation: Maximize objective by choosing the appropriate action
  - Exploration: Gather information on other actions

An optimization interpretation

- Objective (non-concave): $\max_{\theta \in \Theta} \; J(\theta) \; := \; \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} R_t \;\middle|\; \pi_\theta, \mathcal{M}\right]$

- Exploitation: Progress in the gradient direction

$$\theta_{t+1} \; \leftarrow \; \theta_t + \eta_t \widehat{\nabla_\theta J(\theta_t)}$$

- Exploration: Add stochasticity while collecting the episodes

  - noise injection in the action space [28, 16]

$$a = \pi_\theta(s) + \mathcal{N}(0, \sigma^2 I)$$

  - noise injection in the parameter space [23]

$$\tilde{\theta} = \theta + \mathcal{N}(0, \sigma^2 I)$$

Reinforcement learning with Langevin dynamics I

- Explore via an infinite dimensional concave-problem (linear in p):

$$\underset{p \in \mathcal{M}(\Theta)}{\text{maximize}} \quad \mathop{\mathbb{E}}_{\theta \sim p} [J(\theta)]$$

  - $\mathcal{M}(\Theta)$ is the (infinite dimensional) space of all probability distributions on Θ.
  - $p^\star = \arg\max_p \mathop{\mathbb{E}}_{\theta \sim p} [J(\theta)]$ is a delta measure centered at $\theta^\star = \arg\max_\theta J(\theta)$.

Reinforcement learning with Langevin dynamics II

- Exploit via a well-known entropy smoothing trick:

$$\underset{p \in \mathcal{M}(\Theta)}{\text{maximize}} \quad \underset{\theta \sim p}{\mathbb{E}} [J(\theta)] + \beta H(p)$$

  - $H(p) = \underset{\theta \sim p}{\mathbb{E}} [-\log p(\theta)]$ is the entropy of the distribution p.
  - the optimal solution takes the form $p^\star_\beta(\theta) \propto \exp\left(\frac{1}{\beta} J(\theta)\right)$.

- Our proposal for explore-exploit
  - Use Langevin dynamics [36] to draw samples from $p^\star_\beta(\theta)$
  - Use homotopy on the smoothing parameter β

Learning robust policies

- Why robust RL? In short: Generalization under environmental changes
  - upshots: self-driving car in varying environmental conditions
  - trends: from simple parametric models to super expressive neural networks
  - challenges: computational costs as well as the difficulty of training
- Highlight: Robust Adversarial Reinforcement Learning (RARL) [22]
  - train an **agent** neural net
  - train an **adversary** neural net
  - setup a minimax game between the two
- Several variants exist [20, 37, 14, 6, 34, 15].
- Action Robust RL [32].

Two-Player Zero-Sum Markov Game

- Players:
  - Environment: Markov Decision Process (MDP) $\mathcal{M} = (\mathcal{S}, \mathcal{A}, \bar{\mathcal{A}}, T, \gamma, r, \mu)$
  - Agent: parameterized deterministic policy $\pi_\theta : \mathcal{S} \to \mathcal{A}$, where $\theta \in \Theta$
  - Adversary: parameterized deterministic policy $\nu_\omega : \mathcal{S} \to \bar{\mathcal{A}}$, where $\omega \in \Omega$

Two-Player Zero-Sum Markov Game

At time step $t = 0$: $S_0 \sim \mu(\cdot)$

for $t = 1, 2, \ldots$ **do**:

 both players observe the environment's state $S_t \in \mathcal{S}$

 both players choose the actions $A_t = \pi_\theta(S_t) \in \mathcal{A}$, and $\bar{A}_t = \nu_\omega(S_t) \in \bar{\mathcal{A}}$

 the agent gets a reward $R_{t+1} = r(S_t, A_t, \bar{A}_t)$ while the adversary gets $-R_{t+1}$

 both players find themselves in a new state $S_{t+1} \sim T(\cdot \mid S_t, A_t, \bar{A}_t)$

- Performance objective:

$$\max_{\theta \in \Theta} \min_{\omega \in \Omega} J(\theta, \omega) := \mathbb{E}\left[\sum_{t=1}^{\infty} \gamma^{t-1} R_t \,\middle|\, \pi_\theta, \nu_\omega, \mathcal{M} \right]$$

Robust Adversarial Reinforcement Learning (RARL)

- A natural *pure* strategy-based minimax objective

$$\max_{\theta\in\Theta}\min_{\omega\in\Omega} J(\theta,\omega).$$

  - θ: an **agent** neural net
  - ω: an **adversary** neural net
  - *highly* non-concave/non-convex objective

- Theoretical challenges
  - a saddle point might NOT exist [5]
  - no provably convergent algorithm

- Practical challenges
  - the simple (alternating) SGD does NOT work well in practice
  - adaptive methods (Adam, RMSProp,...) highly unstable, heavy tuning

RARL: From pure to mixed Nash Equilibrium

◦ Objective of RARL is a pure strategy formulation:

$$\max_{\theta\in\Theta}\min_{\omega\in\Omega}\; J(\theta,\omega).$$

◦ A new objective of RARL: Our **mixed** strategy proposal via game theory

$$\max_{p\in\mathcal{M}(\Theta)}\;\min_{q\in\mathcal{M}(\Omega)}\; \boldsymbol{E}_{\theta\sim p}\boldsymbol{E}_{\omega\sim q}\left[J(\theta,\omega)\right].$$

where $\mathcal{M}(\mathcal{Z}) \coloneqq \{$all (regular) probability measures on $\mathcal{Z}\}$.

◦ Existence of NE $(p^\star, q^\star)$: Glicksberg's existence theorem [8].

A re-thinking of RARL via the mixed Nash equilibrium

◦ **Upshot:** Our mixed Nash Equilibrium proposal $\equiv$ bi-linear matrix games

$$\max_{p\in\mathcal{M}(\Theta)} \min_{q\in\mathcal{M}(\Omega)} \boldsymbol{E}_{\theta\sim p}\boldsymbol{E}_{\omega\sim q}\left[J(\theta,\omega)\right]$$

$$\Updownarrow$$

$$\max_{p\in\mathcal{M}(\Theta)} \min_{q\in\mathcal{M}(\Omega)} \langle p, Gq\rangle$$

- Caveat: **Infinite dimensions!!!**

◦ Key ingredients moving forward

- $\langle p, h\rangle := \int h dp$ for a measure p and function h (Riesz representation)
- the linear operator G and its adjoint $G^\dagger$:

$$(Gq)(\theta) := \boldsymbol{E}_{\omega\sim q}\left[J(\theta,\omega)\right]$$
$$(G^\dagger p)(\omega) := \boldsymbol{E}_{\theta\sim p}\left[J(\theta,\omega)\right],$$

where $G : \mathcal{M}(\Omega) \to \mathcal{F}(\Theta)$, and $G^\dagger : \mathcal{M}(\Theta) \to \mathcal{F}(\Omega)$.

Training phase

◦ We use the following special adversary with $\alpha = 0.1$ (Noisy Action Robust MDP):

Noisy Action Robust MDP Game

for $t = 1, 2, \ldots$ do:

both players observe the environment's state $S_t \in \mathcal{S}$

both players choose the actions $A_t = \mu(S_t) \in \mathcal{A}$, and $A'_t = \nu(S_t) \in \mathcal{A}$

the resulting action $\bar{A}_t = (1-\alpha)A_t + \alpha A'_t$ is executed in the environment $\mathcal{M}$

the agent gets a reward $R_{t+1} = r(S_t, \bar{A}_t)$ while the adversary gets $-R_{t+1}$

both players find themselves in a new state S_{t+1}

Remarks:

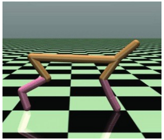
◦ We train the policy based on specific environment parameters

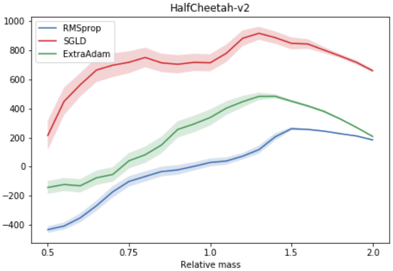
◦ For instance, standard relative mass variables in OpenAI gym.

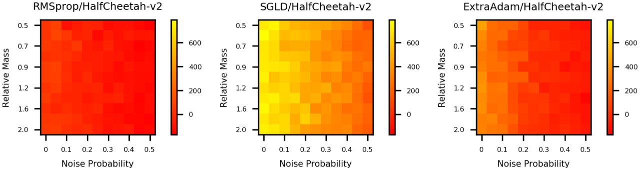
Testing phase

- Robustness under Adversarial Disturbances (x-axis of the heatmap):
  - measure performance in the presence of an adversarial disturbance.
- Robustness to Test Conditions (y-axis of the heatmap):
  - measure performance with respect to varying test conditions.

Experimental evaluation via MuJoCo







Next week!

- Imitation Learning!

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Supplementary: Entropic mirror descent iterates in infinite dimension

- Negative Shannon entropy and its Fenchel dual: (dz :=Lebesgue)
  - $\Phi(p) = \int p \log \frac{dp}{dz}$.
  - $\Phi^\star(h) = \log \int e^h$.
  - $d\Phi$ and $d\Phi^\star$: Fréchet derivatives.[1]

Theorem (Infinite-dimensional mirror descent, informal)

For a learning rate η, a probability measure p, and an arbitrary function h, we can equivalently define

$$p_+ = \mathbf{MD}(p, h) \quad \equiv \quad p_+ = d\Phi^\star\left(d\Phi(p) - \eta h\right) \equiv \quad dp_+ = \frac{e^{-\eta h} dp}{\int e^{-\eta h} dp}.$$

Moreover, most the essential ingredients in the analysis of finite-dimensional prox methods can be generalized to infinite dimension.

- Continuous analog of the entropic mirror descent [2]
  - Mirror-prox also possible [19]

[1] Under mild regularity conditions on the measure/function.

Supplementary: Entropic mirror descent in infinite dimension: rates

◦ Algorithm:

Algorithm 1 Infinite-Dimensional Entropic Mirror Descent

Input: Initial distributions p_1, q_1, and learning rate η
for $t = 1, 2, \ldots, T-1$ **do**
  $p_{t+1} = \mathrm{MD}_\eta\left(p_t, -Gq_t\right)$
  $q_{t+1} = \mathrm{MD}_\eta\left(p_t, G^\dagger p_t\right)$
end for
Output: $\bar{p}_T = \frac{1}{T}\sum_{t=1}^T p_t$ and $\bar{q}_T = \frac{1}{T}\sum_{t=1}^T q_t$

Theorem (Convergence Rates)

Let $\Phi(p) = \int dp \log \frac{dp}{dz}$. *Then*

1. *Entropic MD* $\Rightarrow O(T^{-\frac{1}{2}})$*-NE.*
2. *If only stochastic derivatives (*$\hat{G}^\dagger p$ *and* $-\hat{G}q$*) are available, then Entropic MD* $\Rightarrow O(T^{-\frac{1}{2}})$*-NE in expectation.*